The development of a hyperbolic RANS system for analysing turbomachinery flow field

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ABSTRACT
The hyperbolic form of the Navier-Stokes equations was proposed by H. Nishikawa. It allows the unified discretization of the convective and diffusion fluxes of the equation system, leading to improved accuracy and better convergence. So far the use of the method has been limited to the Navier-Stokes equations without turbulence model equations. For engineering applications, it is imperative to include turbulence modelling in one way or another.

This paper presents the development of a hyperbolic Reynolds averaged Navier-Stokes (RANS) equation with the Spalart-Allmaras turbulence model. With the hyperbolization, both convective flux and diffusion flux can be evaluated using unified schemes. The equation system is solved using a finite volume method, with the fluxes being calculated using an upwind scheme. Time marching is achieved using a four-stage Runge-Kutta method. Local time stepping are also employed to accelerate the solution.

The hyperbolic RANS solver is used to analyze flow field within a compressor cascade, with results being compared to those obtained from a traditional RANS solver. Though the hyperbolic RANS solver demands more memory, which is plenty nowadays and is not issue at all, it has much better convergence and higher accuracy over its traditional counterpart.

INTRODUCTION
In the study of computational fluid dynamics, convection term and dissipation term in Navier-Stokes equations are generally evaluated using different discrete schemes. Most discretizing schemes mainly aim at convection terms. The discrete scheme for dissipation terms is quite scarce, which is usually discretized by the central difference scheme, with only second order accuracy. To change this situation, H. Nishikawa,[1] proposed the first-order hyperbolic method, which enables the dissipation terms to be discretized using the numerical schemes originally designed for the convective terms, leading to unified discretizations.

The first-order hyperbolic method was first applied to the diffusion equations[1], and then extended to Navier-Stokes equations[2]. Later, a number of numerical algorithms based on the hyperbolic form of the Navier-Stokes equations were developed[3,4,5,6,7].

By introducing some new pseudo time partial differential equations, the hyperbolic form of the Navier-Stokes equations eliminates the original second order partial derivative terms in the Navier-Stokes equations, which reduces the order of Navier-Stokes equations from a mixed hyperbolic and parabolic system to a first-order hyperbolic system. The equations of the first-order hyperbolic system are equivalent to the original equations in (pseudo) steady state. The resultant hyperbolic form of the Navier-Stokes equations can be discretized using the same numerical schemes for both convective and dissipation terms.

The first-order hyperbolic system has a couple of advantages. First the gradient quantities are direct solution variables of the newly introduced pseudo time marching equations, which ensures the accuracy of gradient quantities. Hence, the accuracy of variables related to gradient quantities, such as the vorticity, the viscous stresses and the heat fluxes, can also be guaranteed. Second due to the elimination of second derivatives in the Navier-Stokes equations, the first-order hyperbolic system has $O(1/h)$ speed-up[8] during a solution process.

The flow field within turbomachinery is usually turbulent. For practical engineering applications, the unsteady Reynolds averaged Navier-Stokes (RANS) equations are solved to analyse the flow field. The solution of RANS equations requires turbulence modelling. Spalart-Allmaras turbulence model[9] has the advantages of low computational cost and high stability, and is one of the most widely used turbulence models. This paper investigates the hyperbolization of the RANS equations with the Saplalart-Allmaras equation.

METHODOLOGY
In this paper, the hyperbolic RANS system with the Spalart-Allmaras turbulence model equation is constructed according to the HNS20 system[6]. Gradients of velocity, density, temperature and turbulent viscosity coefficient are introduced as solution variables. The two-dimensional
The vector form of the hyperbolic RANS equations can be shown as follows:

\[
\mathbf{p}^{-1} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}
\]  

Where:

\[
\mathbf{U} = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho E \\
\rho u H - \tau_{xx} u - \tau_{yy} v + q_x \\
\rho u y \\
\rho v x \\
\rho v y \\
\rho \nu_{SA} u - \nu_{SA} \\
\nu_{SA}
\end{bmatrix}, \\
\mathbf{F} = \begin{bmatrix}
\rho u \\
\rho u^2 + p - \tau_{xx} \\
\rho uv - \tau_{xy} \\
\rho E \\
\rho u H - \tau_{xx} u - \tau_{yy} v + q_x \\
\rho u y \\
\rho v x \\
\rho v y \\
\rho \nu_{SA} u - \nu_{SA} \\
\nu_{SA}
\end{bmatrix},
\]

\[
\mathbf{G} = \begin{bmatrix}
\rho v \\
\rho u v - \tau_{xx} \\
\rho v^2 + p - \tau_{yy} \\
\rho u H - \tau_{xx} u - \tau_{yy} v + q_x \\
-\nu_{SA} \\
\nu_{SA}
\end{bmatrix},
\]

\[
\mathbf{S} = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho E \\
\rho u H - \tau_{xx} u - \tau_{yy} v + q_x \\
\rho u v - \nu_{SA} + \nu_{SA} \\
\rho v x \\
\rho v y \\
\rho \nu_{SA} u - \nu_{SA} \\
-\nu_{SA}
\end{bmatrix},
\]

\[
\mathbf{p}^{-1} = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho E \\
\rho u H - \tau_{xx} u - \tau_{yy} v + q_x \\
\rho u v - \nu_{SA} + \nu_{SA} \\
\rho v x \\
\rho v y \\
\rho \nu_{SA} u - \nu_{SA} \\
-\nu_{SA}
\end{bmatrix},
\]

\[
diag(1,1,1,1,1,1,1,1,1,1)
\]

The vector in the normal direction of the bounding surface of a control volume, which can be expressed as follows by introducing the unit normal vector \( \mathbf{n} = (n_x, n_y) \):

\[
\mathbf{F}_n = \mathbf{F}_n + \mathbf{G}_n = F_1^i + F_2^i + F_3^i + F_4^i + F_5^i + F_6^i
\]

This shows that the projected flux can be expressed as a superposition of inviscid flux, viscous flux, density flux and
the flux of turbulent viscosity coefficient. These flux components are given as follows in a vector form,

\[
F_n^i = \begin{bmatrix}
\rho u_n \\
\rho u_n u_x + p n_x \\
\rho u_n v + p n_y \\
\rho u_n H \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[F_n^v = \begin{bmatrix}
-\tau_{nx} \\
-\tau_{ny} \\
-\tau_{nv} + q_n \\
-\rho n_x \\
-\rho n_y \\
-\rho n_v \\
\rho \nu_{SA} n_x - \rho k_n \\
\rho \nu_{SA} n_y - \tau_{SA} n_y \\
0 \\
0
\end{bmatrix}
\]

Where:

\[
\tau_{nx} = \tau_{nx} n_x + \tau_{nx} n_y, \tau_{ny} = \tau_{ny} n_x + \tau_{ny} n_y, \\
\tau_{nv} = \tau_{nv} u + \tau_{nv} v, u_n = u n_x + v n_y, q_n = q_n n_x + q_n n_y, \\
r_n = r_n n_x + r_n n_y, k_n = k_n n_x + k_n n_y
\]

(14)

The numerical flux \( \Phi \) is calculated in the framework of the finite volume method, and it can be considered as a sum of the upwind inviscid flux \( \Phi^i \), the upwind viscous flux \( \Phi^v \), the density flux \( \Phi^d \) and the upwind turbulent viscosity coefficient diffusion flux \( \Phi^{sa} \).

\[ \Phi = \Phi^i + \Phi^v + \Phi^d + \Phi^{sa} \]

(15)

Where:

\[ \Phi^i = \frac{1}{2} (F_n^i)_R + (F_n^i)_L - \frac{1}{2} A_n^i \Delta U \]

(16)

\[ \Phi^v = \frac{1}{2} (F_n^v)_R + (F_n^v)_L - \frac{1}{2} P^{-1} PA_n^v \Delta U \]

(17)

\[ \Phi^d = \frac{1}{2} (F_n^d)_R + (F_n^d)_L - \frac{1}{2} P^{-1} PA_n^d \Delta U \]

(18)

\[ \Phi^{sa} = \frac{1}{2} (F_n^{sa})_R + (F_n^{sa})_L - \frac{1}{2} P^{-1} PA_n^{sa} \Delta U \]

(19)

Where \( \Delta U = U_R - U_L \) and \( A_n^i, A_n^v, A_n^d, A_n^{sa} \) is:

\[ A_n^i = \frac{\partial F_n^i}{\partial u}, A_n^v = \frac{\partial F_n^v}{\partial u}, A_n^d = \frac{\partial F_n^d}{\partial u}, A_n^{sa} = \frac{\partial F_n^{sa}}{\partial u} \]

(20)

NUMERICAL RESULTS

The test case used in the paper is a compressor cascade. For the purpose of comparison, the traditional RANS equations based on SA turbulence model are solved together with the hyperbolic RANS equations based on SA turbulence model. For the traditional equation system, the inviscid term is discretized using the first order upwind Roe scheme, and second order central scheme is used to discretize the viscous terms; for the first order hyperbolic RANS system, the first order upwind Roe scheme is used for both inviscid terms and viscous terms. The four stage Runge-Kutta explicit time matching method is used to integrate the two equation systems.

In all computations, we take \( Ma_\infty = 0.5956, Pr = 0.72, \gamma = 1.4, Re_\infty = 1.265 \times 10^5 \) which are used to normalize the equation system.

The mesh of the cascade is shown in figure 1. It has 233 mesh points in the x direction and 73 points in the y direction. At inlet, the total pressure \( P_0 \), the total temperature \( T_0 \) and the flow angle \( \alpha \) is specified; at outlet the back pressure \( P_b \) is specified. Where:

\[ P_0 = 111325 Pa, T_0 = 288.15K, \alpha = 37.5^\circ \]

\[ P_b = 101325 Pa \]

The no-slip and adiabatic conditions are imposed on the blade surface. Periodic boundary conditions are used for the geometric periodic boundary pairs before the blade leading edge region and after the blade trailing edge region.

![Figure 1 Computational grid for two dimensional cascade](image-url)
Fig. 2 shows the residual history of the continuity equation, momentum equations and energy equation from the analyses using both the traditional RANS system and the hyperbolic RANS system. Two points can be summarized from these figures. First, the two methods have almost identical convergence rate at the very beginning of the solution process. This is due to the fact that at the initial solution stage the convergence rate is dictated by the convective process in the flow field. Soon after the initial stage, the hyperbolic system has faster convergence rate. In terms of iterations, the hyperbolic system requires 25% less iterations to reach convergence in comparison with the traditional method.

Figure 3 shows the axial distribution of static pressure coefficient obtained from the two numerical analyses and test. There is small difference on the blade suction side between the two calculations. This difference could be due to the difference in the calculation of viscous terms. The analysis results are in reasonably good agreement with the test data, especially on the blade pressure side.

Figure 4 shows the residual history of the energy equation from the analyses using both the traditional RANS system and the hyperbolic RANS system when the back pressure $P_b$ takes 105325 Pa and other conditions do not change. After 3500 steps, the traditional RANS system diverged, whereas the hyperbolic RANS system could still converge stably. With the increase of the back pressure, the Mach number will decrease, which makes the condition number of the flux Jacobian matrix too large for the original RANS system, while the hyperbolic RANS system can avoid this problem by adjusting the relaxation time. Hence the hyperbolic RANS system is more robust over all speed flows.
CONCLUSIONS

This paper presents the development of a hyperbolic Reynolds averaged Navier-Stokes (RANS) equations with the Spalart-Allmaras turbulence model. A cascade case was used to verify the effectiveness of the proposed method. From the above results, we can draw the following conclusions:

1) When the system is in pseudo steady state, the results obtained by the hyperbolic RANS system are almost the same as those of the traditional methods.

2) Hyperbolic RANS system based on SA turbulence model converges faster than traditional RANS does.

3) As the number of grids increases, the mesh size is reduced and the acceleration effect will become more obvious due to $O(1/h)$ speed-up.

4) When the convergence is dictated by viscous terms, the speed-up will be very obvious. Hence, when the viscous terms are important enough, the speed-up is expected to be $O(1/h)$.

5) Hyperbolic RANS system can reduce the rigidity of the source term about SA turbulence model and increase the stability of the calculation.

6) Hyperbolic RANS system is more robust over all speed flows.

7) Hyperbolic RANS system can reduce the rigidity and has a larger time step.

REFERENCES


